

On In-Plane Current Distribution Producing a Given Axisymmetric Distribution of Normal-to-Plane Magnetic Field

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A very good as well as simple approximate solution is suggested to the problem which flat circular distribution of electric current in finite film produces a given axially symmetric distribution of normal-to-film magnetic field.

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1. In films of type II superconductors, a concentration, n , of vortices piercing the film usually simply correlates with magnetic flux density, \mathbf{B} , at its surface: $\Phi_0 n = (\mathbf{B} - \mathbf{B}_0)_n$, where $(\cdot)_n$ means normal-to-film component of a vector, Φ_0 is magnetic flux quantum, \mathbf{B}_0 - uniform external field, and $\mathbf{B} - \mathbf{B}_0$ is magnetic field induced by (super-)current flowing in the film, \mathbf{J} . The same current stimulates the vortices to move. To consider this motion, analytically or numerically, we should be able to determine what is distribution of \mathbf{J} which produces a given distribution of $\mathbf{B} - \mathbf{B}_0 = \Phi_0 n$.

Such task is non-trivial even under flat geometry, if the film physically occupies a part of its plane and correspondingly the field \mathbf{B} is considered to be known in that part only. Previously, an exact analytical solution of the task was found for the single case of infinitely long flat strip [1, 2], assuming homogeneity of \mathbf{B} along it.

Recently, I suggested exact solution for one more case of round film [3], assuming axial symmetry of \mathbf{B} . The obtained radial distribution of circular current, $J(r)$, as functional of radial distribution of $H(r) \equiv (\mathbf{B} - \mathbf{B}_0)_n$, can be represented by sum of regular term and singular term:

$$J(r) = J_{REG}(r) + J_{SING}(r) , \quad (1)$$

$$J_{REG}(r) = - \int_0^1 T(r, \rho) \rho \frac{dH(\rho)}{d\rho} d\rho , \quad (2)$$

$$J_{SING}(r) = - \frac{2H_e r}{\pi \sqrt{1-r^2}} , \quad (3)$$

$$H_e \equiv - \int_0^1 \frac{H(\rho) \rho d\rho}{\sqrt{1-\rho^2}} \quad (4)$$

Here, J has the sense of the current density, j , multiplied by $2\pi/c$ (in CGS units) and integrated over film's thickness; film's radius is taken to be unit of length; the kernel $T(r, \rho)$ is defined by formulas

$$T(r, \rho) = \frac{2}{\pi} \frac{K(k) - F(\varphi, k) - E(k) + E(\varphi, k)}{\min(r, \rho)} , \quad (5)$$

$$k = \frac{\min(r, \rho)}{\max(r, \rho)} , \quad \varphi = \arcsin \max(r, \rho) ,$$

where standard designations of the complete and incomplete elliptic integrals of 1-st and 2-nd kinds are used (see e.g. [4]). I permit myself to copy from [3] the plots of $T(r, \rho) \rho$ (see Fig.1 below).

Notice, first, that the singular contribution $J_{SING}(r)$ looks exactly as Meissner current in vortex-free film under effective uniform external field H_e . Second, the regular part of the current, $J_{REG}(r)$, turns into zero (or at least remains finite) at film's edge. Thus the condition $H_e = 0$ ensures the absence of any current singularity at the edge. At that, the relation between $J(r)$ and $H(r)$ becomes non-local analogue of the local one, $4\pi j/c = -dH/dr$, which takes place under cylindric geometry.

2. In general, it may be convenient to unify $J_{REG}(r)$ and $J_{SING}(r)$ into the single expression [3]:

$$J(r) = - \frac{d}{dr} \int_0^1 G(r, \rho) \rho H(\rho) d\rho , \quad (6)$$

$$G(r, \rho) = \frac{2}{\pi(\rho+r)} [K(k) - 2F(\phi, k)] , \quad (7)$$

$$k = \frac{2\sqrt{\rho r}}{\rho+r} , \quad \phi = \frac{1}{2} (\arcsin \rho + \arcsin r)$$

Both (5) and (7) contain incomplete elliptic integrals. But, unfortunately, not all of popular mathematical programs can quickly (or let somehow) calculate them. This fact just implies the subject of present article, namely, approximation of the kernel $G(r, \rho)$ in terms of complete elliptic integrals only.

For this purpose, we can apply the series expansion of the kernel $G(r, \rho)$ (see [3]):

$$G(r, \rho) \equiv \frac{2}{\pi} \sum_{k=0}^{\infty} (4k+3) S_k^2 P_{2k+1}(r') P_{2k+1}(\rho') \quad (8)$$

where $x' \equiv \sqrt{1-r^2}$, $\rho' \equiv \sqrt{1-\rho^2}$,

$$S_k \equiv B(1/2, k+1)/2 = 2^k k!/(2k+1)!! , \quad (9)$$

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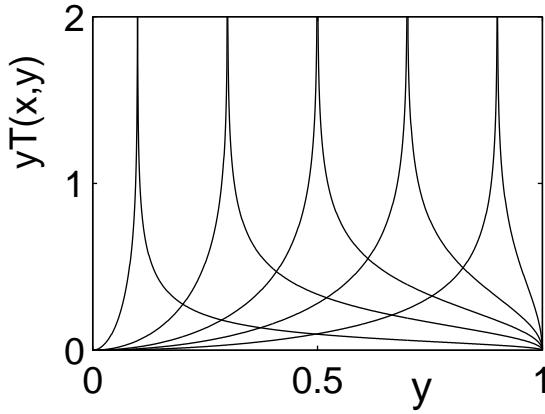


FIG. 1: $yT(x,y)$ via y at $x = 0.1, 0.3, 0.5, 0.7$ and 0.9 .

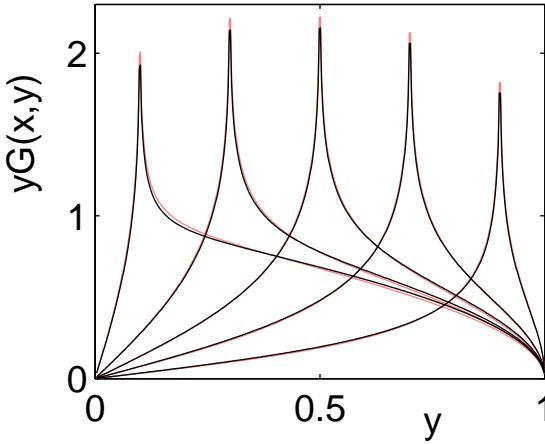


FIG. 2: $yG(x,y)$ (pink curves) in comparison with $yG_{app}(x,y)$ (black curves) via y at $x = 0.1, 0.3, 0.5, 0.7$ and 0.9 .

with $B(x,y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$ being the beta-function, and P_m are ordinary Legendre polynomials [4]. The key to the desired simplification is in the inequalities

$$1 \leq \frac{4k+3}{12} B^2 \left(\frac{1}{2}, k+1 \right) < \frac{\pi}{3} \approx 1.0472, \quad (10)$$

which can be easily proved with the help of the Stirling formula.

Our approximation will be merely replacement of the factor $(4k+3)S_k^2$ by its lower bound, that is by 3, in accordance with (10) and (9). Corresponding approximate series for the kernel $G(r,\rho)$ is

$$G(r,\rho) \approx G_{app}(r,\rho) = \frac{6}{\pi} \sum_{k=0}^{\infty} P_{2k+1}(r') P_{2k+1}(\rho') \quad (11)$$

Fortunately, its sum can be fully expressed via complete elliptic integrals, if use the representation

$$\sum_{k=0}^{\infty} P_{2k+1}(x) P_{2k+1}(y) = \frac{1}{2} [F(x,y) - F(x,-y)],$$

$$F(x,y) \equiv \int_{-\pi}^{\pi} P(e^{i\phi},x) P(e^{-i\phi},y) \frac{d\phi}{2\pi},$$

where $P(t,u)$ is generating function of the Legendre polynomials,

$$P(t,u) = \sum_{n=0}^{\infty} t^n P_n(u) = \frac{1}{\sqrt{1+t^2-2tu}},$$

and then reduce $F(x,y)$ to standard elliptic-type integrals [4, 5]. The result is

$$G_{app}(x,y) = \frac{3\sqrt{2}}{\pi^2} \left[\frac{K(k_-)}{\sqrt{S_-}} - \frac{K(k_+)}{\sqrt{S_+}} \right], \quad (12)$$

$$S_{\pm} \equiv 1 + xy \pm \sqrt{(1-x^2)(1-y^2)}, \quad k_{\pm} \equiv \frac{2xy}{S_{\pm}} \quad (13)$$

Fig. 2 shows this approximation in comparison with formally exact kernel (in your mind please continue the logarithmic peaks to infinity). In practice, relative error of the approximation does not exceed 3 %.

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